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The probabilistic approach for rotating Timoshenko beams

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Abstract

A stochastic modal analysis is developed for reliability studies of rotating beams with uncertain material and sectional properties, uncertain geometric parameters, and random rotating speed. The formulation of the stochastic modal analysis is based on mean-centered second-order perturbation technique and nonlinear eigenvalue analysis. A consistent linearization of the fully geometrically nonlinear beam theory and the virtual work principle are used to derive the governing equations of a rotating Timoshenko beam, and a power series method is employed for the nonlinear eigenvalue analysis to obtain the natural frequencies and vibration modes for free vibration. The effects of variations in material and sectional properties, geometric parameters, and rotating speed on variation in natural frequency are investigated. A sensitivity analysis is performed to identify the important factors on the variation of frequency responses. Here, a definition of resonant failure of a rotating beam is introduced and considered as the limit-state function in the reliability assessment of the rotating beams. To establish the validity of the present probabilistic approach, numerical examples with results obtained by using the Monte Carlo method are given for comparison.

Keywords: Rotating Timoshenko beams; Stochastic modal analysis; Reliability; Nonlinear eigenvalue analysis

1. Introduction

Rotating beams are often used as the simple model for propellers, turbine blades, and satellite booms. The free vibration frequencies of rotating Timoshenko beams have been extensively studied (Wang et al., 1976; Yokoyama, 1988; Lee and Lin; 1994; Du et al., 1994; Nagaraj, 1996). These researches are limited to the deterministic modal or mechanical analysis. However, in reality, components of structural and mechanical systems often exhibit considerable stochastic variations in their properties. Thus, the characteristics of the structure corresponding to these properties also show some stochastic variations. This leads to the necessity to take account of the uncertainties of design parameters in the analysis if highly reliable

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structures are to be designed. In the design and analysis of stochastic dynamical systems, statistical moments of frequency responses are widely used. Boyce and Goodwin (1964) used the perturbation approach for the solution of the eigenvalue problem of random strings and beams. Soong and Bogdanoff (1963) used transfer matrix techniques to investigate the behavior of disordered linear chains. Nakagiri et al. (1985) studied the uncertain eigenvalue analysis of composite laminates by the stochastic finite element method. Hien and Kleiber (1991) discussed the stochastic design sensitivity of the dynamic response. Singh (1985) studied the reliability of turbine blades that predict the probability of the natural frequency and/or the magnifier to exceed the nominal design value. Though much work has been performed in this area, however, the study on the reliability of rotating beams is rather limited in the literature. The combined effects of uncertainties in material and sectional properties, geometric parameters, and rotating speed on the reliability of rotating beams should be studied thoroughly if highly reliable rotating beam structures are to be designed.

The objective of this paper is to investigate the reliability of a rotating beam with random parameters. In this paper, the exact governing equations of rotating Timoshenko beams are derived by consistent linearization of the fully geometrically nonlinear beam theory and the virtual work principle. A power series method is employed for the nonlinear eigenvalue analysis to obtain the natural frequencies and vibration modes for free vibration. Here, it is assumed that failure of a rotating beam called 'resonant failure' will occur when natural frequencies of the rotating beam are located within a range of the specified nominal design values of the rotating speed. The resonant failure of a random rotating beam is used as the limit-state equation in the reliability analysis. A stochastic modal analysis of rotating structures based on the mean-centered second-order method is developed for the reliability analysis. The statistical moments of frequency response of rotating beams with uncertain material and sectional properties, geometric parameters and rotating speed are inferred from the statistics of the base-line random variables (BLRV) through the utilization of the mean-centered second-order method and nonlinear eigenvalue analysis. To establish the validity of the present probabilistic approach, numerical examples are solved and the results using the Monte Carlo method (MCM) are given for comparison.

2. Description of problem

Consider a uniform Timoshenko beam rigidly mounted on the periphery of a rigid hub with radius R which rotates about the hub axis fixed in space at a constant angular speed as shown in Fig. 1. Displacements of the beam are defined in a rotating rectangular Cartesian coordinate system, rigidly tied to the hub. The origin of this coordinate system is chosen to be the intersection of the centroid axes of the hub and the undeformed beam. The X_1 axis is the centroid axis of the undeformed beam, and X_2 and X_3 axes are principal axes of the beam cross-section at the undeformed state. In this paper, all vectors are referred to this coordinate system. The angular velocity of the hub may be given by

$$\Omega = \{0, \Omega \sin\beta, \Omega \cos\beta\}, \tag{1}$$

where the symbol $\{\ \}$ denotes a column matrix, which is used through the paper. The setting angle β of the beam is the angle between the hub axis and the X_3 axis.

Here it is assumed that the beam is only deformed in the X_1–X_3 plane. It is well known that the beam sustains a steady-state axial deformation induced by a constant rotation (Likins, 1973). In this study, the infinitesimally small free vibration of the beam with respect to the position of the steady-state axial deformation and linear stress-strain relationship are considered.

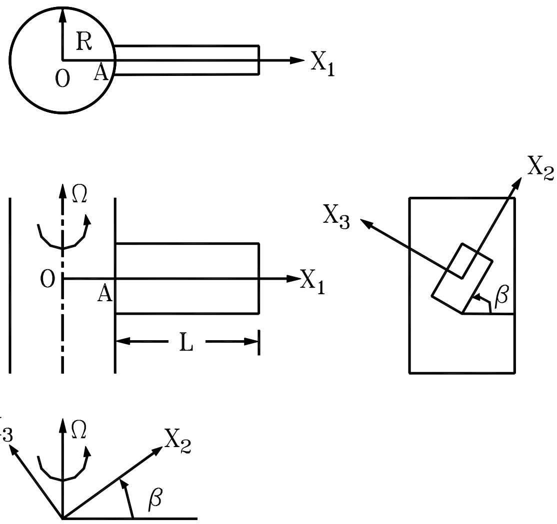

Fig. 1. A rotating Timoshenko beam.

3. Equations of motion

Let P (Fig. 2) be an arbitrary point in the beam element, and Q be the point corresponding to P on the centroid axis. The position vector of point P in the undeformed and deformed configurations may be expressed as

$$\mathbf{r}_0 = \{R + x, y, z\}, \tag{2}$$

$$\mathbf{r} = \{R + x + \bar{u}(x,t) - z\sin\varphi, y, w(x,t) + z\cos\varphi\}, \tag{3}$$

$$\bar{u}(x,t) = u_s(x) + u(x,t), \tag{4}$$

where t is the time, $u_s(x)$ is the steady-state axial deformation induced by constant rotation velocity. During vibration, the infinitesimal displacements of point Q in the X_1 and X_3 directions and the infinitesimal angle of rotation of the cross-section about the negative X_2 axis with respect to the steady-state configuration are $u(x,t)$, $w(x,t)$ and $\varphi(x,t)$, respectively.

The engineering strains of a Timoshenko beam may be approximated by

$$\varepsilon = \varepsilon_0 - z\varphi_{,x}, \tag{5}$$

$$\gamma = w_{,x} - \varphi \tag{6}$$

where ε_0 is the unit extension of the centroid axis, and may be approximated by

$$\varepsilon_0 = \bar{u}_{,x} + \tfrac{1}{2}(\bar{u}_{,x}^2 + w_{,x}^2). \tag{7}$$

The equations of motion for the rotating Timoshenko beam are derived by using the d'Alembert and the virtual work principles. The consistent linearization of fully geometrically nonlinear beam theory is used in the derivation.

Fig. 3 shows a portion of the deformed centerline of the beam. Here the generalized displacements are chosen to be $\bar{u}$, w, and φ defined in Eq. (3). The corresponding generalized forces are F_1, F_3, and M, the

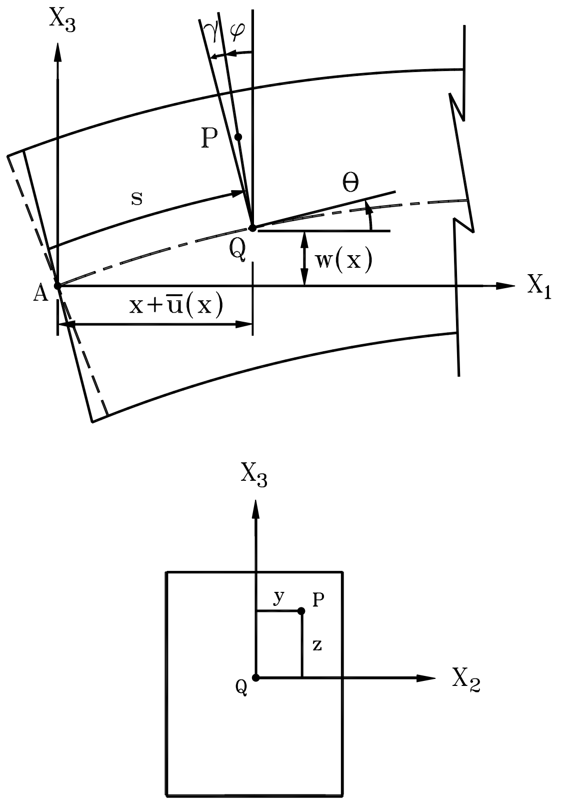

Fig. 2. Kinematics of deformed Timoshenko beam.

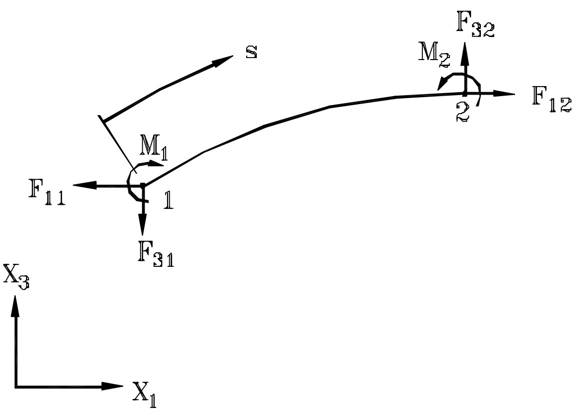

Fig. 3. Free body of a portion of deformed beam.

forces in x_1, x_3 directions, and moment about the negative x_2 axis. F_{1j}, F_{3j}, and M_j $(j = 1, 2)$ in Fig. 3 denote the values of F_1, F_3, and M at jth sections.

The virtual work principle may be written as

$$\delta W_{\text{ext}} = \delta W_{\text{int}} \tag{8}$$

$$\delta W_{\text{ext}} = (F_1 \delta\bar{u} + F_3 \delta w + M\delta\varphi)\Big|_1^2$$
$$= \int_1^2 \frac{\mathrm{d}}{\mathrm{d}x}\Big(M\delta\varphi + F_1\delta\bar{u} + F_3\delta w\Big)\,\mathrm{d}x \tag{9}$$
$$= \int_1^2 \Big(M_{,x}\delta\varphi + M\delta\varphi_{,x} + F_{1,x}\delta\bar{u} + F_1\delta\bar{u}_{,x} + F_{3,x}\delta w + F_3\delta w_{,x}\Big)\mathrm{d}x$$

$$\delta W_{\text{int}} = E\int_{V_{12}} \delta\varepsilon^t\varepsilon\,\mathrm{d}V + \alpha_s\int_{V_{12}}\delta\gamma^t\gamma\,\mathrm{d}V + \rho\int_{V_{12}}\ddot{\mathbf{r}}^t\delta\mathbf{r}\,\mathrm{d}V \tag{10}$$

where δW_{ext} and δW_{int} are the virtual work of the external forces and the internal stresses, respectively, $\delta\varepsilon$ is the variation of ε given in Eq. (5), $\delta\gamma$ is the variation of γ given in Eq. (6), $\delta\mathbf{r}$ is the variation of $\mathbf{r}$ given in Eq. (3), and $\ddot{\mathbf{r}} = \mathrm{d}^2\mathbf{r}/\mathrm{d}t^2$ is the absolute acceleration. In this paper, the symbol $(\dot{\ })$ denotes differentiation with respect to time t. E is Young's modulus, G is the shear modulus, α_s is the shear correction factor, ρ is the density, V_{12} is the volume of the undeformed beam between Sections 1 and 2. The differential volume $\mathrm{d}V$ may be expressed as $\mathrm{d}V = \mathrm{d}A\,\mathrm{d}x$, where $\mathrm{d}A$ is the differential cross-section area of the beam.

The exact expression of δW_{int} may be very complicated. However, due to the assumption of infinitesimal vibration, the quantities u, w, and φ defined in Eqs. (3) and (4), and their derivatives with respect to x and t are all infinitesimal quantities. For linear vibration analysis, only the terms up to the first order of infinitesimal quantities are required. In order to retain all terms up to the first order of infinitesimal quantities in δW_{int}, all terms up to the first order of infinitesimal quantities are retained for $\delta\varepsilon$, ε, $\delta\gamma$, γ, $\delta\mathbf{r}$, and $\ddot{\mathbf{r}}$ in Eq. (10). Note that the steady-state axial deformations $u_s(x)$ in Eq. (4) and its derivatives with respect to x are small finite quantities, not infinitesimal quantities, and are all retained as the zeroth-order terms of infinitesimal quantities.

From Eqs. (3), (5), (6) and (8)–(10), one may obtain

$$AE\bar{u}_{,xx} = \rho A[\ddot{\bar{u}} + \underline{2\dot{w}\Omega\sin\beta} - (R + x + \bar{u})\Omega^2], \tag{11}$$

$$AE(u_{s,x}w_{,x})_{,x} + \alpha_s GA(w_{,xx} - \varphi_{,x}) = \rho A(\ddot{w} - \underline{2\dot{u}\Omega\sin\beta} - w\Omega^2\sin^2\beta), \tag{12}$$

$$EI\varphi_{,xx} = \rho I(\ddot{\varphi} - \varphi\Omega^2\cos^2\beta) - \alpha_s GA(w_{,x} - \varphi), \tag{13}$$

where $I = \int_A z^2\,\mathrm{d}A$ is the moment of inertia of the cross-section, and the underlined terms in Eqs. (11) and (12) are the Coriolis forces.

The boundary conditions for a fixed end at $x = 0$ and for a free end at $x = L$ are given by

$$u_s(0) = u(0,t) = 0, \qquad w(0,t) = 0, \qquad \varphi(0,t) = 0, \tag{14}$$

$$u_{s,x}(L) = u_{,x}(L,t) = 0, \qquad w_{,x}(L,t) - \varphi(L,t) = 0, \qquad \varphi_{,x}(L,t) = 0. \tag{15}$$

4. Nonlinear eigenvalue analysis

Let

$$k = \Omega L\sqrt{\rho/E}. \tag{16}$$

be the dimensionless rotation speed. When $k \ll 1$, the steady-state axial deformation may be approximated by

$$u_s(x) = \frac{\rho\Omega^2}{E}\left(\frac{-x^3}{6} - \frac{Rx^2}{2} + \frac{L^2x}{2} + RLx\right). \tag{17}$$

The vibration of the beam is measured from the position of the steady-state axial deformation. From Eqs. (4), (11)–(13), and (17), the governing equations for free vibration may be expressed as

$$U_{,\xi\xi} = \frac{\rho L^2}{E}[\ddot{U} + \underline{2\dot{W}\Omega\sin\beta} - \Omega^2 U], \tag{18}$$

$$N_{,\xi}W_{,\xi} + NW_{,\xi\xi} + \mu_0(W_{,\xi\xi} - \varphi_{,\xi}) = \frac{\rho L^2}{E}(\underline{-2\dot{U}\Omega\sin\beta} - W\Omega^2\sin^2\beta + \ddot{W}) \tag{19}$$

$$\varphi_{,\xi\xi} = \frac{\rho L^2}{E}(\ddot{\varphi} - \varphi\Omega^2\cos^2\beta) - \mu(W_{,\xi} - \varphi) \tag{20}$$

where

$$\begin{aligned} N &= k^2(-0.5\xi^2 - r\xi + r + 0.5) \\ \mu_0 &= \frac{\alpha_s G}{E}, \quad \mu = \eta^2\mu_0, \quad \eta = \sqrt{AL^2/I}, \end{aligned} \tag{21}$$

$$\xi = \frac{x}{L}, \quad U = \frac{u}{L}, \quad W = \frac{w}{L}, \quad r = \frac{R}{L}, \tag{22}$$

and k is defined in Eq. (16). Note that η is the slenderness ratio of the beam, and N is the steady-state axial strain.

We shall seek a solution of Eqs. (18)–(20) in the form

$$\mathbf{U}(\xi,t) = (\mathbf{U}_R(\xi) + \mathrm{i}\mathbf{U}_I(\xi))\mathrm{e}^{\mathrm{i}\omega t} \tag{23}$$

$$\mathbf{U}(\xi,t) = \{U, W, \varphi\}, \qquad \mathbf{U}_R(\xi) = \{U_R, W_R, \varphi_R\}, \qquad \mathbf{U}_I(\xi) = \{U_I, W_I, \varphi_I\}, \tag{24}$$

where $\mathrm{i} = \sqrt{-1}$, and ω is the natural frequency to be determined. Introducing Eq. (23) into Eqs. (18)–(20), we obtain $U_I = U_R$, $W_R = -W_I$, $\varphi_R = -\varphi_I$, and

$$\mathbf{PA}_{,\xi\xi} + \mathbf{QA}_{,\xi} + \mathbf{RA} = \mathbf{0} \tag{25}$$

$$\mathbf{A} = \{U_R, W_I, \varphi_I\} \tag{26}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & N+\mu_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{,\xi} & -\mu_0 \\ 0 & \mu & 0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} a & d & 0 \\ d & b & 0 \\ 0 & 0 & c \end{bmatrix}, \tag{27}$$

where $a = k^2 + K^2$, $b = K^2 + k^2\sin^2\beta$, $c = -\mu + K^2 + k^2\cos^2\beta$, $d = 2kK\sin\beta$, and K is a dimensionless natural frequency given by

$$K = \omega L\sqrt{\rho/E} \tag{28}$$

It can be seen that Eq. (25) is a set of linear ordinary differential equations with variable coefficients. The solution of Eq. (25) can be expressed as a power series in the independent variable ξ:

$$\mathbf{A}(\xi) = \sum_{n=0}^{\infty} \mathbf{C}_n \xi^n, \tag{29}$$

$$\mathbf{C}_n = \{C_{1n}, C_{2n}, C_{3n}\}, \tag{30}$$

where C_{in} $(i = 1, 2, 3)$ are undetermined coefficients.

Substituting Eq. (29) into Eq. (25) and equating coefficients of like power of ξ, we obtain the recurrence formula

$$\mathbf{C}_n = \mathbf{A}_n \mathbf{C}_{n-2} + \mathbf{B}_n \mathbf{C}_{n-1}, \quad n \geqslant 2,$$

$$\mathbf{A}_n = \frac{-1}{n(n-1)} \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \qquad \mathbf{B}_n = \frac{-1}{n} \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & 0 \end{bmatrix}, \tag{31}$$

where $A_{11} = a$, $A_{12} = d$, $A_{21} = d/f$, $A_{22} = 1/f[b - (n-2)k^2 - 1/2(n-2)(n-3)k^2]$, $A_{33} = c$, $B_{22} = -(n-1)^2 k^2 r/f$, $B_{23} = -\mu_0/f$, $B_{32} = \mu$, in which $f = \mu_0 + k^2(r + 0.5)$, a, b, c, and d are defined in Eq. (27).

From Eq. (31), it can be seen that only $\mathbf{C}_0$ and $\mathbf{C}_1$ are independent constants in Eq. (29), thus, Eq. (29) can be rewritten as

$$\begin{aligned} \mathbf{A}(\xi) &= \left(\mathbf{I} + \sum_{n=2}^{\infty} \xi^n \mathbf{Y}_0^n\right)\mathbf{C}_0 + \left(\xi \mathbf{I} + \sum_{n=2}^{\infty} \xi^n \mathbf{Y}_1^n\right)\mathbf{C}_1 \\ &= \mathbf{E}_1(\xi)\mathbf{C}_0 + \mathbf{E}_2(\xi)\mathbf{C}_1 \end{aligned} \tag{32}$$

where

$$\mathbf{Y}_0^n = \mathbf{A}_n \mathbf{Y}_0^{n-2} + \mathbf{B}_n \mathbf{Y}_0^{n-1}, \qquad \mathbf{Y}_1^n = \mathbf{A}_n \mathbf{Y}_1^{n-2} + \mathbf{B}_n \mathbf{Y}_1^{n-1},$$

$$\mathbf{Y}_0^0 = \mathbf{Y}_1^1 = \mathbf{I}, \quad \mathbf{Y}_0^1 = \mathbf{Y}_1^0 = \mathbf{0}, \tag{33}$$

in which $\mathbf{I}$ and $\mathbf{0}$ are unit matrix and zero matrix of order 3×3, respectively.

From the boundary conditions given in Eqs. (14) and (15), and Eq. (32), one can obtain $\mathbf{C}_0 = \mathbf{0}$ and

$$\mathbf{K}(K)\mathbf{C}_1 = [\mathbf{E}_{2,\xi}(1) - \mathbf{M}\mathbf{E}_2(1)]\mathbf{C}_1 = \mathbf{0}, \tag{34}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mathbf{K}(K)$ denotes $\mathbf{K}$ is function of K given in Eq. (28).

For a nontrivial $\mathbf{C}_1$, the determinant of the 3×3 matrix $\mathbf{K}$ in Eq. (34) must be equal to zero. The values of K which make this determinant vanish are called eigenvalues of matrix $\mathbf{K}$ and give the natural frequencies of the rotating Timoshenko beam through Eq. (28). The bisection method is used here to find the eigenvalues.

5. Uncertainties of the rotating Timoshenko beam

Consider a Timoshenko beam consisting of uniform rectangular cross-section with width H and thickness T. As is well known, the system uncertainties such as material properties (ρ, E and G), sectional parameters (H and T) and geometric parameters (β, R and L) of the rotating beam may fluctuate in the vicinity of the prescribed values during the process of measurement, structural element manufacture, and erection of the structure. Therefore, system parameters should be treated as random rather than deterministic. Besides, rotating beam differs from a nonrotating beam in having additional centrifugal forces and Coriolis effects in its dynamics. The rotating speed Ω of a beam plays an important role in these effects, and may be often varied around the working speed. Thus, the rotating speed is also random in nature.

From now on, b_i $(i = 1, 2, \ldots, 9)$ will be used to denote the BLRV in which b_i $(i = 1, 2, 3)$ are material properties ρ, E and G, respectively; b_i $(i = 4, 5)$ the sectional properties T and H, respectively; b_i $(i = 6, 7, 8)$ geometric parameters β, R and L, respectively; and $b_9 = \Omega$. Since the determination of the interrelations of the BLRV is a difficult task, the adoption of the independence assumption can greatly simplify the reliability assessment. Therefore, the above BLRV b_i are treated to be stochastically independent from each other. Their statistics are used to predict the mechanical behavior of rotating Timoshenko beams.

These uncertainties can cause variations on elements of the matrix $\mathbf{K}$ in Eq. (34). Since the matrix $\mathbf{K}$ is uncertain, the natural frequencies of the rotating beam are also random variates. The statistical moments of the natural frequencies may be derived from those of the base-line random parameters in the mean-centered second-order perturbation approach.

6. The stochastic modal analysis

The most adopted technique for analyzing the stochastic response of random systems in engineering is the mean-centered second-order approach. The major advantage of this technique is that the multivariate distribution functions need not to be known but only the first two moments. One of the inherent limitations of the second-order technique is that the uncertainties cannot be large, i.e., variances of the random variables cannot be large when compared with their mean. Typically, the coefficient of variation (COV) is around 10% although it has been shown that it could be as high as 20% for acceptable results to be obtained (Ang and Tang, 1984; Cornell, 1972). The basic idea of this second-order approach is to expand the random response about the mean value of BLRV and to retain only up to second-order terms. To derive the formulation of stochastic modal analysis, the frequency response ω of a rotating beam may be expanded in terms of the random variables b_i $(i = 1, 2, \ldots, M)$ as

$$\omega = \omega^{(0)} + \sum_{i=1}^{M} \omega_{,i}^{(1)} \Delta b_i + \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{M} \omega_{,ij}^{(2)} \Delta b_i \Delta b_j \tag{35}$$

where $\Delta b_i = b_i - \overline{b}_i$ with $\overline{b}_i$ denoting the mean value of the random variable b_i; $M = 9$; $\omega^{(0)}$ is the zeroth-order term of natural frequency, which is identical to the deterministic natural frequency; $\omega_{,i}^{(1)}$ is the first-order term of natural frequency with respect to random variables b_i; $\omega_{,ij}^{(2)}$ is the second-order term of natural frequency with respect to random variables b_i and b_j.

The zero-order term of natural frequency $\omega^{(0)}$ may be determined by solving the nonlinear eigenvalue problem of Eq. (34) using the bisection method as mentioned. The expressions of elements in $\mathbf{K}(\omega)$ and normality condition on the eigenvectors, respectively, in terms of BLRV are complicated and difficult. The analytical solution for the derivatives of nonlinear eigenvalues (Jankovic, 1988) may be difficult to achieve. Thus, the following finite difference approximation is employed here to calculate the sensitivity terms of the natural frequency of rotating beams.

$$\omega_{,i}^{(1)} = \left. \frac{\omega(b_i + \Delta b_i) - \omega(b_i)}{\Delta b_i} \right|_{\underline{b} = \underline{\overline{b}}} \tag{36}$$

$$\omega_{,ii}^{(2)} = \left.\frac{\omega(b_i + \Delta b_i) - 2\omega(b_i) + \omega(b_i - \Delta b_i)}{(\Delta b_i)^2}\right|_{\underline{b}=\overline{\underline{b}}} \tag{37}$$

The routine DERIV of IMSL (1989) mathematical package is employed to evaluate the first- and second-order terms of natural frequency statistics.

Based on the mean-centered second-order method (Ang and Tang, 1984), the approximate mean involves the second-order effect while the approximate variance is only the first order. They are given as follows:

$$E[\omega] \cong \omega^{(0)} + \frac{1}{2}\sum_{i=1}^{9}\sum_{j=1}^{9}\omega_{,ij}^{(2)}\,\mathrm{Cov}(b_i, b_j) \tag{38}$$

$$\mathrm{Var}[\omega] \cong \sum_{i=1}^{9}\sum_{j=1}^{9}\omega_{,i}^{(1)}\omega_{,j}^{(1)}\,\mathrm{Cov}(b_i, b_j) \tag{39}$$

where $\mathrm{Cov}(b_i, b_j)$ is the covariance between b_i and b_j.

When the BLRV are stochastically independent, $\mathrm{Cov}(b_i, b_j) = 0$, $i \neq j$, and $\mathrm{Cov}(b_i, b_j) = \mathrm{Var}[b_i]$, $i = j$. Thus Eqs. (38) and (39) may be rewritten as

$$E[\omega] \cong \omega^{(0)} + \frac{1}{2}\sum_{i=1}^{9}\omega_{,ii}^{(2)} S_{b_i}^2 \tag{40}$$

$$S_{\omega}^2 \cong \sum_{i=1}^{9}(\omega_{,i}^{(1)})^2 S_{b_i}^2 \tag{41}$$

where S_{ω} and S_{b_i} are the standard deviation of ω and b_i, respectively.

7. Sensitivity analysis

In general, sensitivity analysis is an important part of stochastic structural analysis. One benefit of sensitivity studies is the identification of the design parameters that have the greatest effect on the estimated statistics of structural responses. The other benefit is the ability to identify those parameters that can be taken as fixed values and need not be considered as random variables in the corresponding stochastic model. Efficiency of most stochastic structural analysis methods depends strongly on the number of random variables. It is really helpful to be able to model a component or system with a small number of random variables. In this paper, the effects of uncertainties of the base-line random parameters on the variation of frequency responses are studied using the following sensitivity measure.

$$\eta_i = \frac{\partial S_{\omega}}{\partial S_{b_i}}\frac{S_{b_i}}{S_{\omega}} \tag{42}$$

where η_i is defined as the relative sensitivity coefficient which is a measure of the fractional change of S_{ω} with respect to a fractional change in S_{b_i}; ∂ is the differential operator. In view of Eq. (42), the above equation can be rewritten as

$$\eta_i = \omega_{,i}^{(1)2}\frac{S_{b_i}^2}{S_{\omega}^2} \tag{43}$$

Based on the above equation, the BLRV that have significant effects on the variation of natural frequency can be identified.

8. Reliability analysis

As is well known, a machine operating within a range of frequencies near its natural frequency will undergo high vibration. Here, it is assumed that a structural failure called 'resonant failure' will occur when natural frequencies of the rotating beam are located within a range of the specified nominal design values of the rotating speed. In this paper, the resonant failure of the rotating beam is considered in the reliability analysis. The resonant failure probability may be written as

$$P_f = \mathrm{Prob}[E[\Omega](1-\chi) \leqslant \omega \leqslant E[\Omega](1+\chi)] \tag{44}$$

where $\chi = 0.05\zeta$ and ζ is a constant parameter for ranging the failure bound.

The reliability assessment of a rotating beam structure, in general, requires information on the probability distribution and not just statistical moments of the natural frequency of the structure. In Section 7, however, only statistical moments of natural frequency can be determined, while the types of probability distributions of the natural frequencies are indeterminate. As smooth probability distribution functions commonly used by practicing engineers produce reasonable results for reliability assessments of mechanical/structural components, here, the lognormal distribution denoted by $f_\omega(u)$ is assumed for the probability density functions of ω. Thus the probability in Eq. (44) can be written as

$$P_f = \int_a^b f_\omega(u)\,\mathrm{d}u \tag{45}$$

where $a = E[\Omega](1-\chi)$, $b = E[\Omega](1+\chi)$.

The probability of Eq. (45) can be evaluated by the numerical integration. The Monte-Carlo simulation will be employed to verify the accuracy of the present reliability assessment of the random rotating beam subject to the resonant failure.

9. Numerical examples

To demonstrate the accuracy and efficiency of the present deterministic formulation, firstly, numerical examples for the natural frequencies of a rotating Timoshenko beam are studied. Here cases with and without considering the Coriolis force, referred to as Case A and Case B, respectively, are considered, and the corresponding results are referred as to Present A and Present B, respectively.

In order to compare present results with those reported in the literature, the dimensionless natural frequency $\Lambda = \eta K = \omega L^2\sqrt{\rho A/EI}$ and dimensionless rotational speed $\alpha = \eta k = \Omega L^2\sqrt{\rho A/EI}$ are employed here. The natural frequencies of the rotating beams for different values of the setting angle are listed in Table 1. As expected, for $\beta = 0$ the results of Present A, Present B and those reported in the literature are in close agreement. For $\beta = 90°$, the discrepancies between the results of Present A and Present B are noted. This observation indicates that the effect of the Coriolis force on the natural frequencies of the rotating Timoshenko beam is not negligible. Therefore, even though it is observed that the effect of the Coriolis force on the natural frequencies decreases for higher modes, the Coriolis force is considered in the following stochastic modal analysis and the corresponding reliability assessment of the rotating beam. To study natural frequency statistics of the rotating Timoshenko beam with different BLRV, the following cases are studied: Case I, only material properties were treated as random; Case II, only rotating speed was random; Case III, only sectional properties were random; Case IV, only geometric properties were random. Since

Table 1
Dimensionless frequencies for the rotating Timoshenko beam ($\alpha = 10$, $r = 3$, $1/\mu = 3.0588\eta$)

β	$\eta = 0.01$				$\eta = 0.0025$			
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_1	Λ_2	Λ_3	Λ_4
0								
A	22.938	44.781	66.287	71.967	23.491	55.984	96.913	143.71
B	22.938	44.781	66.287	71.967	23.491	55.984	96.913	143.71
C	23.050	45.598	67.716	73.076	23.524	56.105	97.188	144.49
D	22.938	44.781	66.287	71.967	23.491	55.984	96.913	143.71
E	23.037	45.428	66.854	72.313	23.514	56.072	97.011	143.82
90								
A	8.500	29.152	49.372	74.141	16.691	37.751	91.041	140.98
B	20.853	44.957	66.677	71.985	21.298	55.240	96.594	143.50
C	20.867	45.115	67.520	72.756	21.313	55.284	96.747	144.21
D	20.753	44.315	66.109	71.620	21.277	55.162	96.473	143.43
E	20.850	44.955	66.668	71.982	21.302	55.250	96.570	143.53

A: Present A, B: Present B, C: given by Yokoyama (1988), D: given by Lee and Lin (1994), E: given by Wang et al. (1976).

existing alternative solutions are limited, the MCM is used to obtain results for comparison. Without the loss of generality, here the first two natural frequencies of the beam are investigated. Unless stated, otherwise, the statistics of BLRV given in Table 2 are used in the following numerical studies.

The COV for natural frequencies denoted by V_{ω_i} $(i = 1, 2)$ is investigated first. Results for different BLRV obtained by using the present and MCMs are obtained. The effects on COV correspond to various $E[\Omega]$ for $\beta = 0°$ for the first and second modes shown in Figs. 4 and 5. It is noted that the results obtained by the present method are in excellent agreement with those obtained by the MCM in which over 5000 data have been generated for each case. As can seen from Fig. 4 for the setting angle $\beta = 0°$ among different BLRV, the randomness of sectional properties has greater effects on V_{ω_1} as $E[\Omega] \leqslant 2000$ rpm and the randomness of rotating speed has greater effects as $E[\Omega] > 2000$ rpm. It can be seen that V_{ω_1} corresponding to the random rotating speed monotonically increases with the increase of $E[\Omega]$ and V_{ω_1} corresponding to the other randomness decreases with the increase of $E[\Omega]$. Fig. 5 shows the V_{ω_2} of the second natural frequency for the beam with different BLRV. For $\beta = 90°$ with different BLRV, the trend of variation of COV with respect to $E[\Omega]$ is very much similar for the first and second modes. Results for the first mode are shown in Fig. 6 for illustration. It can be seen that the randomness of sectional properties may have the greater effect on the V_{ω_i} $(i = 1, 2)$, and, as expected, the COV of the first two natural frequencies,

Table 2
Statistics of the BLRV

BLRV	Mean	Standard deviation
ρ	11.79 g/cm^3	1.18 g/cm^3
E	165.44 GPa	16.54 GPa
G	63.44 GPa	6.34 GPa
T	4.0 cm	0.4 cm
H	1.5 cm	0.15 cm
β		1.8°
L	80.0 cm	2.4 cm
R	8.0 cm	0.8 cm
Ω		10%[a]

[a] Data presents the COV of BLRV.

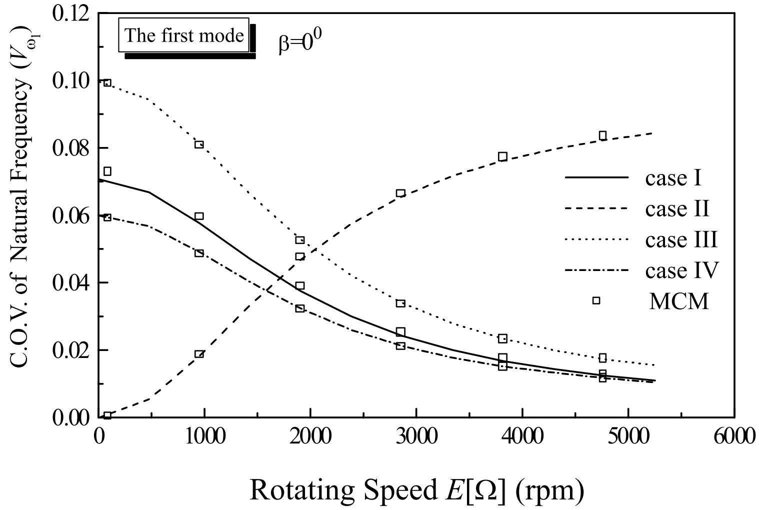

Fig. 4. The COV of the first natural frequency vs. the expected value of rotating speed for the Timoshenko beam with different random variables ($\beta = 0°$).

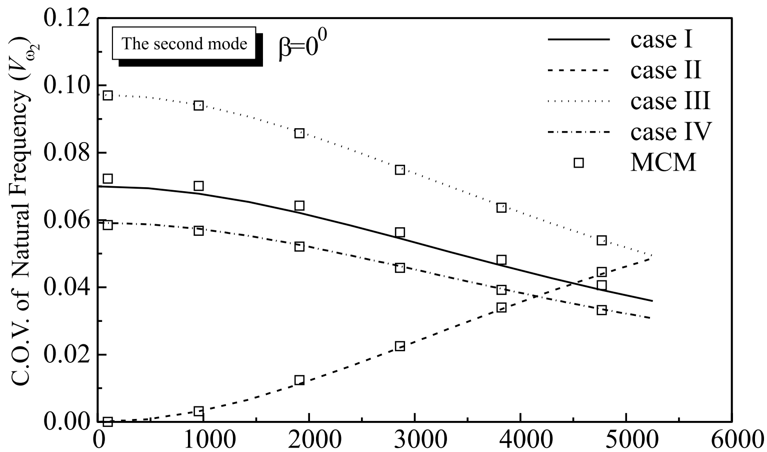

Fig. 5. The COV of the second natural frequency vs. the expected value of rotating speed for the Timoshenko beam with different random variables ($\beta = 0°$).

($V_{\omega_i}, i = 1, 2$), corresponding to the random rotating speed have the increasing influence by the increase of $E[\Omega]$.

Figs. 7 and 8 show the mean values and three standard-deviation bounds of the frst two natural frequencies for the beam for $\beta = 0°$ and $\beta = 90°$, respectively, with all BLRV. In Figs. 7 and 8, the relations between plus or minus three standard-deviation bounds, $\pm 3S_\omega$, of natural frequencies and the expected value of rotating speed, $E[\Omega]$, are constructed. The results obtained by the present method are in excellent agreement with those obtained by the MCM in which over 5000 data have been generated for each case.

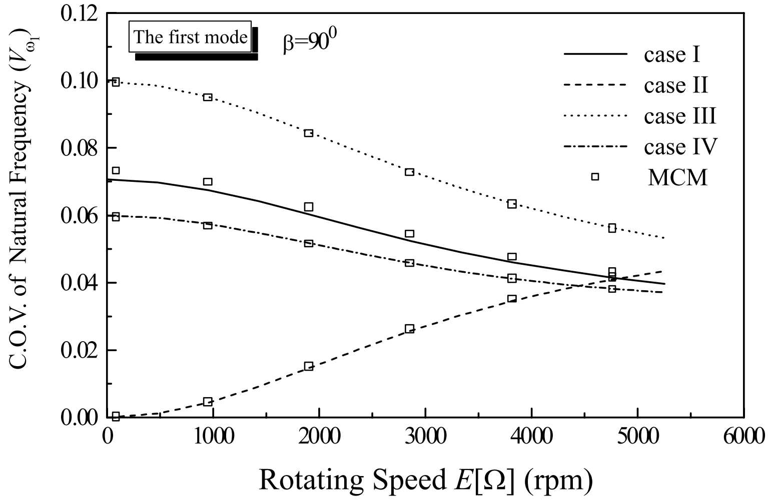

Fig. 6. The COV of the first natural frequency vs. the expected value of rotating speed for the Timoshenko beam with different random variables ($\beta = 90°$).

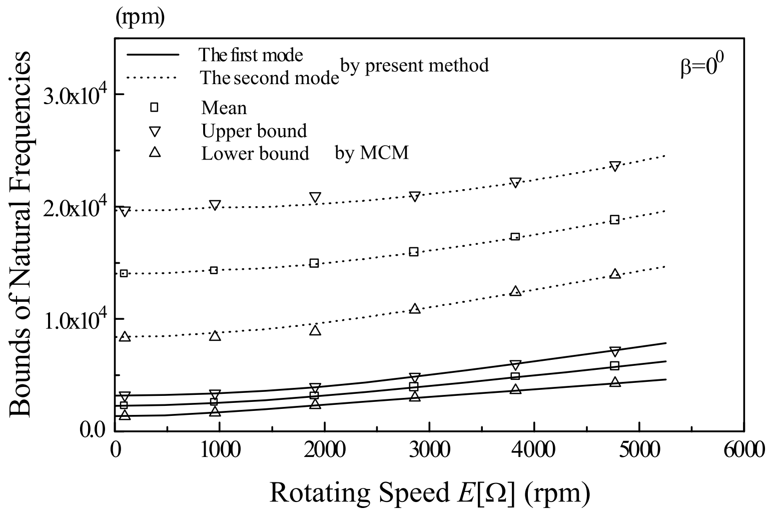

Fig. 7. Plus or minus $3S_\omega$ bounds of the natural frequencies vs. the expected value of rotating speed for the Timoshenko beam with all random variables ($\beta = 0°$).

In order to investigate the important factor of the BLRV on the variation of natural frequencies, the sensitivity analysis of the frequency responses will be studied in advance. The relative sensitivity coefficients derived from Eq. (43) for $\beta = 0°$ shown in Figs. 9 and 10 for first two natural frequencies of the beam, respectively. As the trend of variation for $\beta = 90°$ case is similar for the first and second modes, results for the first mode only are shown in Fig. 11. It is noted that the random parameters ρ, E, T, L and Ω have more significant effects on the variation of the natural frequencies for the rotating beams than the other random parameters. Irrelevant to the setting angles $\beta = 0°$ and $\beta = 90°$, the random variable T has the most effect

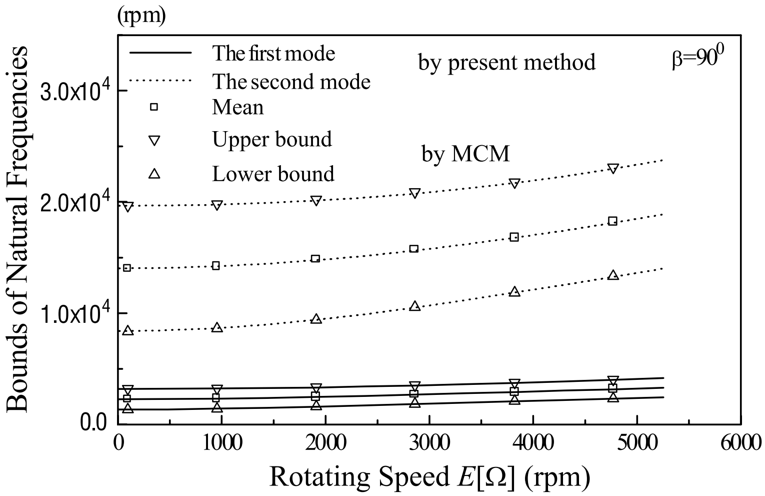

Fig. 8. Plus or minus $3S_{\omega}$ bounds of the natural frequencies vs. the expected value of rotating speed for the Timoshenko beam with all random variables ($\beta = 90°$).

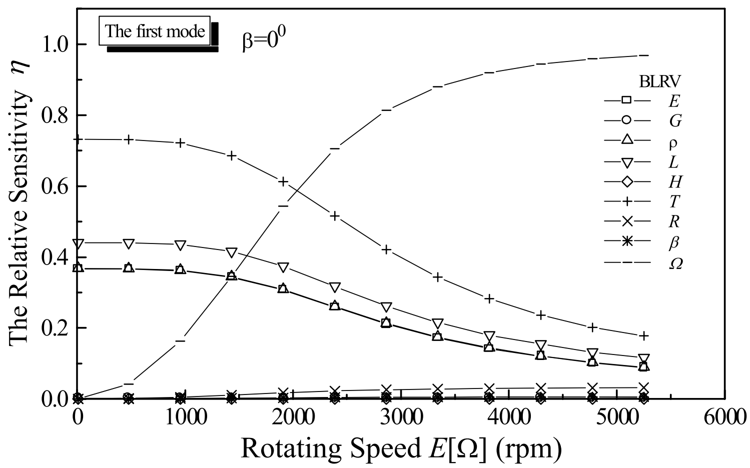

Fig. 9. The sensitivities of the first natural frequency vs. the expected value of rotating speed for the Timoshenko beam with the setting angle $\beta = 0°$.

on the variation of the natural frequencies and, as expected, the effect of Ω will increase on the variation of the natural frequencies as the expected value of rotating speed is increased. Therefore, it is important to have tight control on the sectional thickness and rotating speed if higher reliability for the rotating beams is desired. It is also worth noting that the random parameters G, H, R and β have relative small η, thus, these random parameters may be treated as constant (deterministic) in the corresponding stochastic modal analysis.

Finally, the reliability of the rotating beams with all BLRV subject to the resonant failure will be studied. The definitions of P_f was given in Eq. (45). It is noted that in Figs. 12 and 13 the results obtained by the present method are in excellent agreement with those obtained by the MCM in which over 10 000 data have

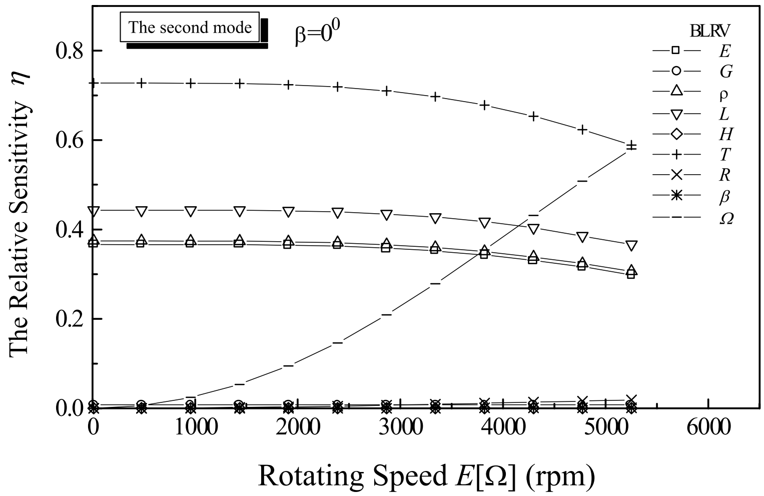

Fig. 10. The sensitivities of the second natural frequency vs. the expected value of rotating speed for the Timoshenko beam with the setting angle $\beta = 0°$.

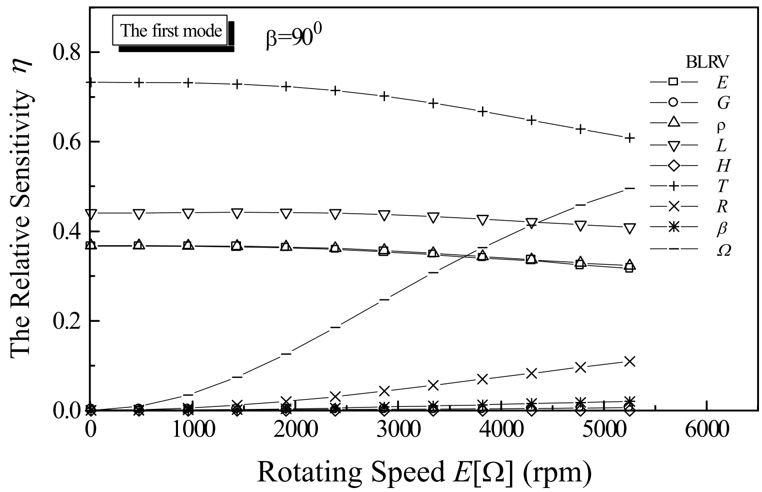

Fig. 11. The sensitivities of the first natural frequency vs. the expected value of rotating speed for the Timoshenko beam with the setting angle $\beta = 90°$.

been generated for each case. It should be noted that the effect of the failure bound ζ will play the important role on the reliability assessment.

10. Conclusions

The randomness of frequency responses of rotating Timoshenko beams with random parameters has been investigated by the stochastic modal analysis based on the mean-centered second-order perturbation

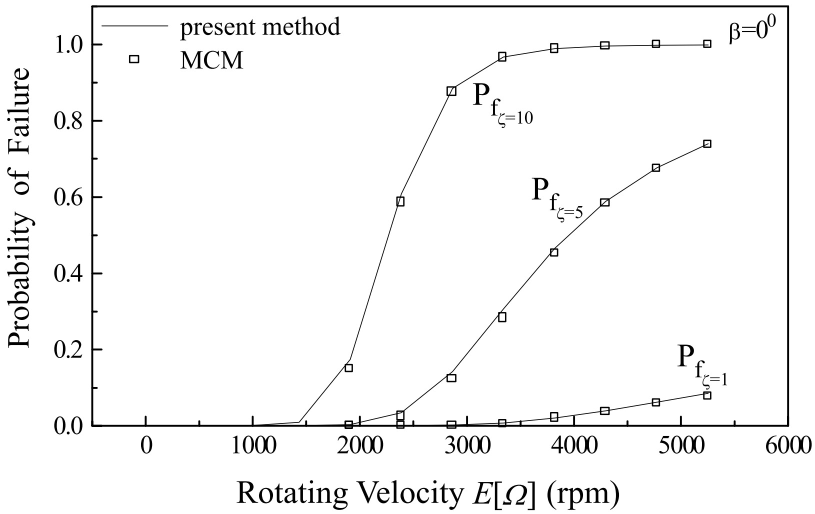

Fig. 12. Failure probability of the Timoshenko beam with all random variables and the setting angle $\beta = 0°$ for different failure bounds.

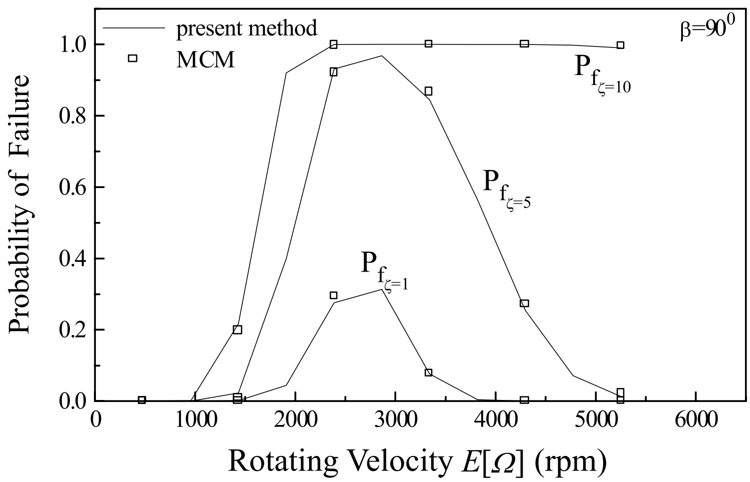

Fig. 13. Failure probability of the Timoshenko beam with all random variables and the setting angle $\beta = 90°$ for different failure bounds.

technique and nonlinear eigenvalue analysis. The feasibility and accuracy of the present approach are validated by comparing present results with those given in the existing literature using deterministic modal analysis, and to results obtained via the MCM in the stochastic modal analysis. In the present study, a sensitivity analysis of the variation of frequency responses was also performed. It has been found that sectional thickness and rotating speed have significant effects on the variation of frequency responses. A tight control on these parameters is required if high reliability on the design of rotating Timoshenko beams is desired. It was also noting that the random parameters such as shear modulus, width of rectangular cross-section, radius of the hub, and the setting angle had relative insignificant effects on variation of frequency responses, therefore, those random parameters may be able to be treated as constants in the stochastic modal analysis. In addition, the reliability of the rotating beams with all BLRV subject to the resonant failure are studied. Based on the present probabilistic analysis, it is believed that the procedure may be valuable for developing the reliability analysis of rotating beams.

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